Integral of One Over X to the Fourth Plus One

Brian Rogers

August 14, 2007

We wish to solve the following integral:

$$\int \frac{1}{x^4+1}\, dx$$

To proceed, we will first convert this into a form which is suitable for partial fraction decomposition.

$$\begin{aligned}
\frac{1}{x^4+1} &= \frac{1}{x^4+2x^2+1-2x^2} \\
&= \frac{1}{(x^2+1)^2-2x^2} \\
&= \frac{1}{\left(x^2+1-\sqrt{2}x\right)\left(x^2+1+\sqrt{2}x\right)}
\end{aligned}$$

We now have a factored polynomial in the denominator with two irreducible quadratic terms. We can proceed with partial fraction decomposition:

$$\begin{aligned}
\frac{1}{x^4+1} &= \frac{Ax+B}{x^2+1-\sqrt{2}x} + \frac{Cx+D}{x^2+1+\sqrt{2}x} \\
1 &= (Ax+B)\left(x^2+1+\sqrt{2}x\right) + (Cx+D)\left(x^2+1-\sqrt{2}x\right) \\
1 &= Ax^3+Ax+A\sqrt{2}x^2+Bx^2+B+B\sqrt{2}x+Cx^3+Cx-C\sqrt{2}x^2+Dx^2+D-D\sqrt{2}x \\
1 &= (A+C)x^3+(A\sqrt{2}+B-C\sqrt{2}+D)x^2+(A+B\sqrt{2}+C-D\sqrt{2})x+(B+D)
\end{aligned}$$

Since there are no x^3 terms in the left side of the equation we know immediately that $A+C=0$. Similarly, it must be true that $B+D=1$:

$$1 = (A\sqrt{2}-C\sqrt{2}+1)x^2+(B\sqrt{2}-D\sqrt{2})x+1$$

We can now write four equations:

$$\begin{aligned}
A\sqrt{2}-C\sqrt{2} &= -1 \\
B\sqrt{2}-D\sqrt{2} &= 0 \\
A+C &= 0 \\
B+D &= 1
\end{aligned}$$

Thus, $A=-\sqrt{2}/4$, $B=1/2$, $C=\sqrt{2}/4$, $D=1/2$. We now have two integrals:

$$\int \frac{-x\sqrt{2}/4+1/2}{x^2+1-\sqrt{2}x}\, dx + \int \frac{x\sqrt{2}/4+1/2}{x^2+1+\sqrt{2}x}\, dx$$

With some strategic factoring, we can get an expression in each numerator that is more conducive to u-substitution:

$$-\sqrt{2}/8 \int \frac{2x - 2\sqrt{2}}{x^2 + 1 - \sqrt{2}x}\ dx + \sqrt{2}/8 \int \frac{2x + 2\sqrt{2}}{x^2 + 1 + \sqrt{2}x}\ dx$$

We can start with the integral on the left:

$$-\sqrt{2}/8 \int \frac{2x - 2\sqrt{2}}{x^2 + 1 - \sqrt{2}x}\ dx$$

Let $u = x^2 + 1 - \sqrt{2}x$ so that $du = \left(2x - \sqrt{2}\right)\ dx$. This gives us two integrals:

$$\begin{aligned} -\sqrt{2}/8 \int \frac{du}{u} + 1/4 \int \frac{dx}{x^2 + 1 - \sqrt{2}x} &= \\ -\sqrt{2}/8 \ln|u| + 1/4 \int \frac{dx}{x^2 + 1 - \sqrt{2}x} &= \\ -\sqrt{2}/8 \ln\left|x^2 + 1 - \sqrt{2}x\right| + 1/4 \int \frac{dx}{x^2 + 1 - \sqrt{2}x} &= \\ -\sqrt{2}/8 \ln\left(x^2 + 1 - \sqrt{2}x\right) + 1/4 \int \frac{dx}{x^2 + 1 - \sqrt{2}x} & \end{aligned}$$

To take care of the other integral, we can complete the square on the bottom to prepare for an arctan-like integral:

$$\begin{aligned} 1/4 \int \frac{dx}{x^2 + 1 - \sqrt{2}x} &= \\ 1/4 \int \frac{dx}{\left(x - \sqrt{2}/2\right)^2 + 1/2} &= \\ 1/4 \int \frac{dx}{\left(x - \sqrt{2}/2\right)^2 + \left(\sqrt{2}/2\right)^2} &= \\ 1/4 \cdot \sqrt{2} \arctan\left(\sqrt{2}\left(x - \sqrt{2}/2\right)\right) &= \\ \sqrt{2}/4 \arctan\left(x\sqrt{2} - 1\right) & \end{aligned}$$

We are halfway done. The other integral from above can be solved in a very similar way:

$$\sqrt{2}/8 \int \frac{2x + 2\sqrt{2}}{x^2 + 1 + \sqrt{2}x}\ dx$$

Let $u = x^2 + 1 + \sqrt{2}x$ so that $du = \left(2x + \sqrt{2}\right)\ dx$. This gives us two integrals:

$$\begin{aligned} \sqrt{2}/8 \int \frac{du}{u} - 1/4 \int \frac{dx}{x^2 + 1 + \sqrt{2}x} &= \\ \sqrt{2}/8 \ln|u| - 1/4 \int \frac{dx}{x^2 + 1 + \sqrt{2}x} &= \\ \sqrt{2}/8 \ln\left|x^2 + 1 + \sqrt{2}x\right| - 1/4 \int \frac{dx}{x^2 + 1 + \sqrt{2}x} &= \\ \sqrt{2}/8 \ln\left(x^2 + 1 + \sqrt{2}x\right) - 1/4 \int \frac{dx}{x^2 + 1 + \sqrt{2}x} & \end{aligned}$$

Once again, complete the square on the bottom for the remaining integral:

$$\begin{aligned}
-1/4\int\frac{dx}{x^2+1+\sqrt{2}x} &= \\
-1/4\int\frac{dx}{\left(x+\sqrt{2}/2\right)^2+1/2} &= \\
-1/4\int\frac{dx}{\left(x+\sqrt{2}/2\right)^2+\left(\sqrt{2}/2\right)^2} &= \\
-1/4\cdot\sqrt{2}\arctan\left(\sqrt{2}\left(x+\sqrt{2}/2\right)\right) &= \\
-\sqrt{2}/4\arctan\left(x\sqrt{2}+1\right)
\end{aligned}$$

Our final answer is thus

$$\sqrt{2}/8\ln\left(x^2+1+\sqrt{2}x\right)-\sqrt{2}/8\ln\left(x^2+1-\sqrt{2}x\right)+\sqrt{2}/4\arctan\left(x\sqrt{2}-1\right)-\sqrt{2}/4\arctan\left(x\sqrt{2}+1\right)$$